

# SANDWICH THEOREMS FOR SHIODA–INOSE STRUCTURES

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ABSTRACT. We give a geometric construction of the three infinite series of K3 surfaces which are sandwiched by Kummer surfaces within a Shioda–Inose structure. Explicit examples are also provided.

## 1. INTRODUCTION

Shioda–Inose structures have recently featured very prominently in the arithmetic and geometry of K3 surfaces, relating specific K3 surfaces to Kummer surfaces in a natural way. In [10], Shioda showed that any jacobian elliptic K3 surface with two singular fibres of type  $II^*$  is in fact sandwiched by the Kummer surface in question (which is indeed of product type). Subsequently Ma gave an abstract Hodge theoretic proof that any Shioda–Inose structure can be extended to a sandwich [6]. However, for the generic situation of Picard number 17 there are only five explicit geometric examples due to Kumar [5], van Geemen–Sarti [2] and Koike [4].

In this paper we will develop three infinite series of K3 surfaces of Picard number (at least) 17 with a sandwiched Shioda–Inose structure by geometric means:

**Theorem 1.** *Let  $N \in \mathbb{N}$ . Assume that one of the following three alternatives holds:*

- (1)  $\forall p \mid N : p \equiv 1 \pmod{4}$ ;
- (2)  $N = \prod_i p_i$  or  $N = 7 \prod_i p_i, \forall i : p_i \equiv 1, 2, 4 \pmod{7}$ ;
- (3)  $N = \prod_i p_i$  or  $15 \prod_i p_i$  for an odd number of primes  $p_i \equiv 2, 8 \pmod{15}$   
or  
 $N = 3 \prod_i p_i$  or  $5 \prod_i p_i$  for an even number of primes  $p_i \equiv 2, 8 \pmod{15}$ .

*Then for any K3 surface  $X$  with a primitive embedding  $T_X \hookrightarrow U^2 + \langle -2N \rangle$  there is an explicit geometric sandwiched Shioda–Inose structure.*

The theorem will be proved by exhibiting three distinct families of K3 surfaces in Sections 3–5. See 3.5, 4.4, 5.4 for the precise arguments. Our construction

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uses specialisation from 4-dimensional families of K3 surfaces via lattice enhancements and elliptic fibrations with 2-torsion sections. We point out that our construction in particular allows us to realise all five previously known examples (by inspection of the discriminants, see Section 2), but it does not give any transcendental lattices  $U^2 + \langle -2N \rangle$  beyond those specified in Theorem 1 (cf. 3.6).

The paper is organised as follows. The next section reviews basics on Shioda–Inose structures and sandwiches. Each of the subsequent sections is devoted to one of the families in question. Throughout we work over the field  $\mathbb{C}$  of complex numbers although the equations provided will make perfect sense over any field  $k$  of characteristic different from 2.

## 2. SHIODA–INOSE STRUCTURES AND SANDWICHING

A classical example of K3 surfaces consists in Kummer surfaces: starting from an abelian surface  $A$ , we consider the quotient by inversion with respect to the group structure. This attains 16 rational double point singularities (type  $A_1$ ) which can be resolved to a K3 surface denoted by  $\text{Km}(A)$ . Thus there is a rational map of degree 2

$$A \dashrightarrow \text{Km}(A).$$

This is also reflected in the transcendental lattices, i.e. the orthogonal complements  $T_X$  of  $\text{NS}(X)$  inside  $H^2(X, \mathbb{Z})$  with respect to cup-product. Namely the transcendental lattices are similar, i.e. the rank is constant while the intersection form is multiplied by 2:

$$T_{\text{Km}(A)} = T_A(2).$$

From the classification point of view, a natural problem is to relate  $\text{Km}(A)$  to a K3 surface  $X$  with the original transcendental lattice

$$T_X = T_A.$$

This was first achieved by Shioda and Inose in [11] in the case of product type  $A \cong E \times E'$  where  $E, E'$  are elliptic curves. Their construction (geared towards K3 surfaces with Picard number 20) makes crucial use of jacobian elliptic fibrations on  $\text{Km}(E \times E')$ . In fact,  $X$  is shown to admit a rational map to  $\text{Km}(E \times E')$  of degree 2. In other words  $X$  admits a Nikulin involution (8 isolated fixed points) whose quotient gives rise to  $\text{Km}(E \times E')$  as its resolution. Following Morrison [7], exactly this is usually required as ingredient of a Shioda–Inose structure:

$$(1) \quad \begin{array}{ccc} A & & X \\ & \searrow \quad \swarrow & \\ & \text{Km}(A) & \end{array}$$

In the situation of non-isogenous elliptic curves, one has  $\text{NS}(A) = U$  and  $T_A = U^2$ , where  $U$  denotes the hyperbolic plane  $\mathbb{Z}^2$  with intersection form  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

The generic situation is somewhat different as  $A$  will have Picard number one. If  $A$  is endowed with a polarisation of degree  $2N$  ( $N \in \mathbb{N}$ ), then we have

$$T_A = U^2 + \langle -2N \rangle.$$

In terms of lattice theory Morrison gave a complete answer which K3 surfaces admit a Shioda–Inose structure [7]:

**Theorem 2** (Morrison). *An algebraic K3 surface  $X$  admits a Shioda–Inose structure if and only if there is a primitive embedding*

$$T_X \hookrightarrow U^2 + \langle -2N \rangle \quad \text{for some } N \in \mathbb{N}.$$

An equivalent criterion is that  $X$  admits a (Nikulin) involution interchanging two orthogonal copies of  $E_8$  in  $\text{NS}(X)$ , the unique unimodular even negative-definite lattice of rank 8. Or even more abstractly:  $E_8^2 \hookrightarrow \text{NS}(X)$ .

In order to discuss sandwiching, we return to the product type situation  $A = E \times E'$  from [11] that we alluded to above. In [10] Shioda noticed that this case comes automatically with a sandwich. Namely  $\text{Km}(E \times E')$  itself possesses a Nikulin involution which gives rise to  $X$ , thus extending the diagram (1) as follows:

$$(2) \quad \begin{array}{ccc} & & \text{Km}(E \times E') \\ & & \swarrow \text{dashed} \\ E \times E' & & X \\ & \searrow \text{dashed} & \\ & \text{Km}(E \times E') & \end{array}$$

This brings us to the problem whether every Shioda–Inose structure can be extended to a sandwich. An affirmative answer was given recently by Ma in [6]:

**Theorem 3** (Ma). *Any Shioda–Inose structure admits a sandwich.*

We emphasise that Ma’s proof is Hodge theoretic in nature; in particular it does not give any geometric information. Indeed there are only five  $N \in \mathbb{N}$  in terms of Theorem 2 for which an explicit geometric construction has been exhibited:

$N = 1$	due to Kumar [5]	– Kummer surfaces of jacobians of genus 2 curves
$N = 2$	van Geemen–Sarti [2]	
$N = 3, 5, 7$	due to Koike [4]	– elliptic K3 surfaces with MW rank zero

In each case the quotients are constructed through elliptic fibrations with a 2-torsion section, so that in fact  $X$  and  $\text{Km}(A)$ , interpreted as elliptic curves over  $k(t)$ , are related by an isogeny. This will also be our preferred approach in the sequel. Namely we will construct 3-dimensional families of elliptic fibrations with MW-rank 1 and 2-torsion sections exhibiting a Shioda–Inose structure. As stated in Theorem 1, this construction allows us to realise an infinite series of families of K3 surfaces with sandwich structure including the five families

known before. For two cases of small  $N$  missing from the above list of examples ( $N = 4, 8$ ) we will also provide explicit equations for the families.

### 3. FIRST SERIES

Our starting point is a 4-dimensional family of K3 surfaces whose generic member  $X$  has

$$\mathrm{NS}(X) = U + E_7^2.$$

This family which has also recently been investigated in [1], can be given as an elliptic fibration with two singular fibres of Kodaira type  $III^*$  at 0 and  $\infty$ :

$$(3) \quad X : y^2 = x^3 + t^3 a(t)x + t^5 b(t), \quad a(t), b(t) \in k[t] \text{ of degree } 2.$$

Here one can still rescale  $(x, y)$  and separately  $t$  to normalise 2 coefficients. The hyperbolic plane  $U \subset \mathrm{NS}(X)$  is spanned by the zero section  $O$  and the general fibre  $F$  while the  $E_7$ 's comprise fibre components disjoint from  $O$ .

The family is a natural starting point since it specialises to Kumar's family with  $\mathrm{NS} = U + E_7 + E_8$  by setting, for instance,  $\deg(a) \leq 1$ . The transcendental lattice of  $X$  can be computed with Nikulin's theory of the discriminant form (or as a 2-elementary lattice) as

$$(4) \quad T_X = U^2 + A_1^2.$$

Here we consider the dual lattice  $L^\vee$  of a non-degenerate even integral lattice  $L$ . It gives rise to the discriminant group  $L^\vee/L$ , a finite abelian group of size the square of the discriminant of  $L$ . The discriminant group comes with an induced quadratic form from  $L$  which is denoted by

$$q_L : L^\vee/L \rightarrow \mathbb{Q}/2\mathbb{Z}.$$

Presently we have isomorphisms of abelian groups

$$E_7^\vee/E_7 \cong A_1^\vee/A_1 \cong \mathbb{Z}/2\mathbb{Z}.$$

In fact there are generators of square  $-3/2$  resp.  $-1/2$  which give a direct identification

$$(5) \quad q_{E_7} \cong -q_{A_1}.$$

**3.1. Lattice enhancement.** A convenient way to specialise our 4-dimensional family of K3 surfaces to a subfamily of Picard number  $\rho \geq 17$  is to enhance  $L = \mathrm{NS}$  by some vector  $v$  of  $T_X$  of negative square. In this context the only subtlety consists in the primitive closure in the K3 lattice  $\Lambda$ :

$$L' := \overline{\langle L, v \rangle} \subset \Lambda = H^2(X, \mathbb{Z}) = U^3 + E_8^2.$$

Then the theory of lattice polarised K3 surfaces guarantees that  $L'$  corresponds to a 3-dimensional (sub)family consisting of K3 surfaces with generically  $\mathrm{NS} = L'$  and transcendental lattice  $T' = v^\perp \subset T_X$ . Since we are interested in transcendental lattices containing two copies of  $U$  by Theorem 2, we will always enhance  $L$  with a vector from  $A_1^2$  in the sequel.

*Example 4.* Take the vector  $v = (1, 0) \in A_1^2$ , i.e. a generator of a single  $A_1$ . Via the isomorphism (5),  $v/2$  corresponds to a generator  $w$  of  $E_7^\vee/E_7$ . Thus we obtain

$$L' := \overline{\langle L, v \rangle} = \langle L, v/2 + w \rangle.$$

In fact, we have just glued together one copy of  $E_7$  and  $A_1$  each to the unimodular lattice  $E_8$ , so  $L' = U + E_7 + E_8$ .

Since lattice enhancements involve the primitive closure, we have to consider which integers  $A_1^2$  represents primitively, i.e. by a vector  $v = (v_1, v_2)$  with  $\gcd(v_1, v_2) = 1$ . An easy exercise in quadratic forms gives

**Lemma 5.**  $A_1^2$  represents  $-2N$  primitively if and only if  $N$  is a product of primes  $\equiv 1 \pmod{4}$  or twice such.

We have already discussed the lattice enhancement for  $N = 1$  in Example 4. The other cases correspond to  $v_1 v_2 \neq 0$  and not both even. Let  $w_1, w_2$  denote the generators of  $E_7^\vee/E_7$  of square  $-3/2$ , matching those for the  $A_1$ 's via (5). Then we find

$$L' := \overline{\langle L, v' \rangle} \quad \text{with} \quad v' = v/2 + \begin{cases} w_1 & v_1 \text{ odd} \\ 0 & v_1 \text{ even} \end{cases} + \begin{cases} w_2 & v_2 \text{ odd} \\ 0 & v_2 \text{ even} \end{cases}$$

Note that according to our set-up  $v'$  is indeed an integral even vector:

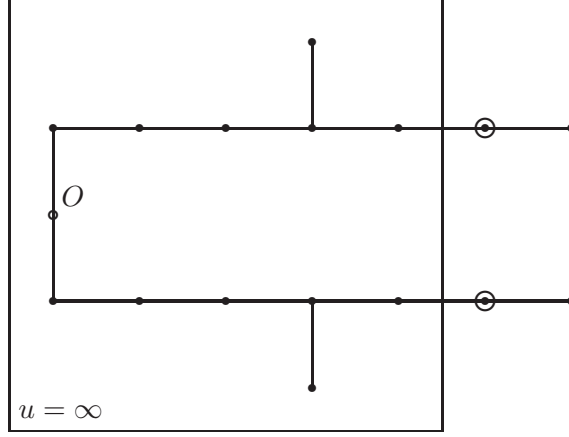
$$2\mathbb{Z} \ni v'^2 = -N/2 - \begin{cases} 3/2 & N \text{ odd} \\ 3 & N \text{ even} \end{cases}$$

If  $N \neq 1$ , then the additional algebraic class  $v'$  can directly be identified with a section  $P$  made orthogonal to  $O$  and  $F$  in NS while meeting one or both  $III^*$  fibres in the non-identity component depending on the parity of  $v_1$  and  $v_2$ . In terms of the theory of Mordell-Weil lattices [9], the contributions from  $w_1, w_2$  to  $v'^2$  correspond to correction terms for those fibre components. In consequence  $P$  has height  $h(P) = N/2$ .

**3.2. Transcendental lattice.** By construction the subfamily with  $L' \hookrightarrow \text{NS}$  has generically transcendental lattice  $T' = U^2 + \langle -2N \rangle$ .

**3.3. Alternate elliptic fibration.** We shall now switch to an alternative elliptic fibration on  $X$  which comes with a 2-torsion section. For this purpose we identify a divisor  $D$  of Kodaira type  $I_8^*$  supported on  $O$  and the components of the  $III^*$  fibres as depicted in the next figure.

The linear system of this divisor  $D$  will induce another elliptic fibration  $X \rightarrow \mathbb{P}^1$ . Here the rational curves adjacent to  $D$  will serve as zero section and 2-torsion section. The latter claim is easily verified with the height pairing after realising that the two remaining components of the original  $III^*$  fibres (disjoint from  $I_8^*$ ) generically sit in two fibres of type  $I_2$ . Other than these, the new fibration has generically 6 fibres of type  $I_1$ .

FIGURE 1.  $I_8^*$  Divisor supported on two  $III^*$ 's and  $O$ 

Explicitly the divisor  $D$  can be extracted by the parameter  $u = x/t^2$  with respect to (3). Elementary transformations give the Weierstrass form

$$(6) \quad X : y^2 = t(u^3 t^2 + u a(t) + b(t)),$$

an elliptic curve over  $k(u)$  with 2-torsion section  $(0,0)$ . Upon lattice enhancement, the rational curve  $P$  gives a multisection for the alternate fibration which induces a section  $P'$  again of height  $h(P') = N/2$ .

**3.4. Kummer surface.** Consider the 2-isogenous elliptic surface  $Y$  arising from the alternate elliptic fibration on  $X$  by quotienting by translation by  $(0,0)$  and resolving singularities. Generically  $Y$  has singular fibres  $I_4^*, 2 \times I_1, 6 \times I_2$  on top of the 2-torsion section. The transcendental lattice can be computed as  $T_Y = U(2)^2 + A_1^2$ . Now we consider the specialisation  $Y'$  of  $Y$  corresponding to the lattice enhancement of  $X$  yielding the section  $P$  (and  $P'$ ).

**Proposition 6.** *Let  $N$  be an odd integer as in Lemma 5. Then  $Y'$  is a Kummer surface with  $T_{Y'} = T_X(2) = U(2)^2 + \langle -4N \rangle$ .*

*Proof.* The property of being a Kummer surface follows from the 2-divisibility of  $T_{Y'}$ , giving an even lattice of rank at least 17 as in Theorem 2. We shall now prove that the transcendental lattice takes exactly the shape as stated.

The enhanced section  $P'$  pulls back to a section  $P^*$  on  $Y$  of height  $h(P^*) = 2h(P') = N$ . We claim that this section is not 2-divisible in  $\text{MW}(Y')$ . Otherwise there were a section  $Q$  with  $2Q = P$ , so  $h(Q) = N/4$ . But then, the correction terms in the height pairing are all half integers by inspection of the present singular fibres. Hence  $h(Q) \in \frac{1}{2}\mathbb{Z}$ , giving the required contradiction.

It follows that  $\text{NS}(Y')$  is generated by fibre components, zero and 2-torsion sections together with  $P^*$ . In particular we compute by [8, (22)]

$$(7) \quad \text{disc NS}(Y') = \frac{4 \times 2^6}{2^2} h(P^*) = 2^6 N.$$

Then we use that push-forward induces an embedding

$$T_X(2) \hookrightarrow T_{Y'}$$

(which need not be primitive in general, but both lattices have the same rank by [3]; see [11], [7]). By (7) both lattices have the same discriminant, hence they agree.  $\square$

*Remark 7.* If  $N$  is even, then quite on the contrary the section  $P^*$  becomes 2-divisible in  $\text{MW}(Y')$  (by inspection of the 2-length, compare Remark 11). Hence the discriminants of  $T_X(2)$  and  $T_{Y'}$  do not match. In fact, the two cases of lattice enhancements from Lemma 5 are swapped: odd  $N$  on  $X$  corresponds to  $2N$  on  $Y$ , but even  $N$  on  $X$  gives odd  $N/2$  on  $Y$ .

**3.5. Proof of Theorem 1 (1).** Let  $N$  be composed of primes  $\equiv 1 \pmod{4}$  as in Theorem 1 (1). Let  $X'$  be the family of K3 surfaces given by the lattice enhancement of  $X$  corresponding to  $N$ . Then any  $X'$  with  $\rho(X') = 17$  fits into a sandwiched Shioda–Inose structure by Proposition 6 using the 2-isogeny and its dual. Thus we only have to rule out that the construction degenerates for higher Picard number.

Note that any non-degenerate member of the family comes with the corresponding elliptic fibration with 2-torsion section (possibly with different singular fibres). It remains to show that the quotient does always have 2-divisible transcendental lattice, regardless of the Picard number. This follows from the next lemma.  $\square$

**Lemma 8.** *Let  $(X, \iota)$  be a family of K3 surfaces with Nikulin involution  $\iota$ . Assume that the resolution  $Y$  of the quotient  $X/\iota$  generically has transcendental lattice*

$$T_Y = T_X(2).$$

*Then the same applies to any non-degenerate specialisation of  $(X, \iota)$ .*

*Proof.* Let  $(X', \iota')$  be a non-degenerate specialisation with quotient  $Y'$ . Naturally these come with primitive embeddings

$$T_{X'} \hookrightarrow T_X, \quad T_{Y'} \hookrightarrow T_Y.$$

Using the push-forward via the quotient map and the assumption from the lemma, we obtain the diagram

$$\begin{array}{ccc} T_{Y'} & \hookrightarrow & T_Y \\ \cup & & \parallel \\ T_{X'}(2) & \hookrightarrow & T_X(2) \end{array}$$

Going through the lower right corner, we find that  $T_{X'}(2)$  embeds primitively into  $T_Y$ . Arguing with the upper left corner, we deduce the same for the inclusion  $T_{X'}(2) \subset T_Y$ . Since these lattices have the same rank by [3], we deduce equality.  $\square$

**3.6. Optimality of construction.** The alert reader might wonder whether we might not be able to derive more K3 surfaces from our construction by not limiting ourselves to the summand  $A_1^2$  of  $T$  in Lemma 5. Indeed the transcendental lattice  $T$  does represent any integer (in many ways). However, recall that in essence we aim at primitive embeddings

$$(8) \quad U^2 + \langle -2N \rangle \hookrightarrow T.$$

Here we explain that our construction does not give rise to any subfamilies with such transcendental lattice other than those stated in Theorem 1. For this purpose, we first embed the summand  $U^2$  into  $T$ . In general, the orthogonal complement  $M = (U^2)^\perp \subset T$  need not be unique, but for sure the genus of  $M$  is fixed by  $T$ . Presently, the genus of  $M$  consists of a single lattice by class group theory. Hence in order to study primitive embeddings (8) it does indeed suffice to consider representations of  $-2N$  by  $A_1^2$ . Verbatim, the same argument will go through for the other two constructions (see Lemma 10, 13).

**3.7. Other lattice enhancements.** For the sake of completeness, we briefly comment on other lattice enhancements of the family  $X$ . This will also serve as a sanity check of the above argument. Analogous arguments apply to the K3 families in the next two sections.

Let  $X'$  be a family of K3 surfaces of Picard number 17 and discriminant  $2N$  ( $N > 1$ ) that arises from  $X$  by lattice enhancement. As before,  $X'$  comes with an additional section  $P$  of height  $N/2$ , say with respect to the original elliptic vibration. Consider the case where  $N$  is odd. Then  $P$  meets some non-identity fibre components, but the section  $2P$  of height  $2N$  does not. Let  $\varphi$  denote the orthogonal projection with respect to  $O, F$  in  $\text{NS}(X')$ . By assumption  $\varphi(2P)$  is orthogonal to the trivial lattice (i.e. to the image of  $\text{NS}(X)$  in  $\text{NS}(X')$ ), and we have

$$\varphi(2P)^2 = -2N, \quad \varphi(2P) \cdot \varphi(P) = -N.$$

In particular we find that  $\frac{1}{N}\varphi(2P) \in \text{NS}(X')^\vee$ . In the discriminant group  $\text{NS}(X')^\vee / \text{NS}(X')$  this induces an element of order  $N$  where the discriminant form evaluates as  $-2/N$ .

Now assume that  $T(X') = U^2 + \langle -2N \rangle$  is one of the transcendental lattices in question. Its discriminant form also evaluates as  $-2/N$  at an element of order  $N$ . But then for  $\text{NS}(X')$  and  $T(X')$  to be orthogonal complements in the K3 lattice, we require that their discriminant forms have reversed signs. In particular this implies that  $-1$  is a square modulo any prime divisor of  $N$ . Thus we find exactly the conditions of Theorem 1.

For  $N$  even, we distinguish two further cases by the parity of  $M = N/2$ . If  $N/2$  is odd, then essentially the same argument as above goes through with the element  $\frac{1}{M}\varphi(P) \in \text{NS}(X')^\vee$  of order  $N$  in the discriminant group. If  $N/2$  is even, then  $P$  is in the narrow Mordell-Weil lattice, meeting all singular fibres at the identity component. Thus  $\text{NS}(X') = \text{NS}(X) + \langle \varphi(P) \rangle$ , and the discriminant group has 2-length 3 exceeding that of  $U^2 + \langle -2N \rangle$ , contradiction.



## 4. SECOND SERIES

In this and the next section, we shall argue directly with elliptic fibrations with 2-torsion section, this time semi-stable. Such fibrations admit an extended Weierstrass form

$$y^2 = x(x^2 + a(t)x + b(t))$$

with reducible fibres at the zeros of  $b(t)$ . We treat two families which lend themselves directly to Shioda–Inose structures. In this section, we consider surfaces with singular fibres of Kodaira type  $I_{14}, I_2$ . The corresponding K3 surfaces form a 4-dimensional family

$$(9) \quad X : y^2 = x(x^2 + a(t)x + t)$$

where  $a \in k[t]$  has degree 4 and does not vanish at  $t = 0$ . Generically  $\rho(X) = 16$ .

**4.1. Transcendental lattice.** The transcendental lattice can be read off directly from an alternative elliptic fibration on  $X$ . Namely it is easy to extract a divisor of Kodaira type  $II^*$  from  $I_{14}$  extended by  $O$  and the identity component of  $I_2$ . The remaining components of  $I_{14}$  give rise to a section and an  $A_6$  configuration of rational curves. Comparing ranks (and discriminants), we find generically

$$\mathrm{NS}(X) = U + A_6 + E_8 \quad \text{and} \quad T_X = U^2 + \begin{pmatrix} -2 & -1 \\ -1 & -4 \end{pmatrix}.$$

The latter representation is easily verified with the discriminant form since  $A_6^\vee/A_6$  has a generator of square  $-6/7$ .

**4.2. Quotient family.** Consider the quotient by translation by the 2-torsion section and denote the resulting elliptic K3 surfaces by  $Y$ . Then generically  $Y$  has singular fibres  $I_7, I_1, 8 \times I_2$  and  $\mathrm{MW} \cong \mathbb{Z}/2\mathbb{Z}$ .

**Lemma 9.** *Generically  $Y$  has transcendental lattice  $T_Y \cong T_X(2)$ .*

*Proof.* By [3]  $Y$  also has Picard number 16, so the MW-rank is 0 by the Shioda–Tate formula. Standard formulas exclude any further torsion. Hence  $\mathrm{NS}(Y)$  is generated by fibre components and the two torsion sections; in particular  $\mathrm{NS}(Y)$  has discriminant  $2^6 7$ . Then, as before, the push-forward embedding  $T_X(2) \hookrightarrow T_Y$  is an isometry.  $\square$

**4.3. Lattice enhancement.** As before we enhance  $\mathrm{NS}(X)$  by a vector from the last summand of  $T_X$ . Generally we find the following possibilities:

**Lemma 10.**  $\begin{pmatrix} -2 & -1 \\ -1 & -4 \end{pmatrix}$  represents  $-2N$  primitively if and only if  $N$  is a product of primes  $\equiv 1, 2, 4 \pmod{7}$  or seven times such a product.

Now pick a vector  $v$  as in Lemma 10 and enhance NS by a generator of  $v^\perp$  in the above rank 2 lattice. We infer the transcendental lattice

$$T' = U^2 + \mathbb{Z}v = U^2 + \langle -2N \rangle.$$

On the elliptic fibrations this implies an additional singular fibre ( $N = 1$ ) or a section  $P$  of height  $2N/7$  ( $N > 1$ ).

**4.4. Proof of Theorem 1 (2).** Let  $N$  be as in Theorem 1 (2) (or equivalently Lemma 10). Let  $X'$  be a member of the subfamily of K3 surfaces given by the lattice enhancement of  $X$  corresponding to  $N$  as above. Then by assumption  $T_{X'}$  embeds primitively into  $U^2 + \langle -2N \rangle$ . Moreover the quotient surface has transcendental lattice  $T_{X'}(2)$  by Lemma 8 in conjunction with Lemma 10. Using the 2-isogeny and its dual, these surface realise a sandwiched Shioda–Inose structure.  $\square$

*Remark 11.* The above construction implies that the image of the section  $P$  on the quotient surface becomes 2-divisible. This fact can also be checked directly on the quotient surface by an argument comparing the 2-length of NS to the rank of the transcendental lattice.

**4.5. Example:**  $N = 4$ . We need to endow the elliptic fibration (9) with a section of height  $8/7$ . Up to translation by the 2-torsion section and inversion, this is uniquely achieved by a section  $P$  meeting  $I_{14}$  at  $\Theta_4$  (the fourth component) and  $I_2$  at the identity component, but not intersecting  $O$ . Indeed the height pairing gives  $h(P) = 4 - \frac{4 \cdot 10}{14} = \frac{8}{7}$ .

With the present extended Weierstrass form (9),  $P$  can only take the shape  $(\alpha, w)$  with  $\alpha \in k^*$  and  $w \in k[t]$  of degree 2. But then the pair  $(\alpha, w)$  uniquely determines the polynomial  $a(t)$  in (9) by

$$w^2 = \alpha^3 + \alpha^2 a(t) + \alpha t.$$

Hence we obtain the 3-dimensional family of K3 surfaces with generically  $T = U^2 + \langle -8 \rangle$  (and the unirationality of the corresponding moduli space).

## 5. THIRD SERIES

As third series we treat elliptic K3 surfaces with 2-torsion section and singular fibres of type  $I_{10}, I_6$ . Here the Weierstrass form can be transformed to

$$(10) \quad X : y^2 = x(x^2 + a(t)x + t^3),$$

again yielding a 4-dimensional family with  $\rho = 16$  generically. The arguments are very similar to the previous family, so we only outline the main ideas.

**5.1. Transcendental lattice.** With the discriminant form one finds the transcendental lattice generically as

$$T_X = U^2 + \begin{pmatrix} -4 & -1 \\ -1 & -4 \end{pmatrix}.$$

The computations may be eased by switching to another elliptic fibration, for instance with fibres of type  $III^*$  and  $IV^*$  and a section of height  $5/2$  (elliptic parameter  $u = x/t$ ). This gives a representation of  $q_{\text{NS}(X)}$  as  $\mathbb{Z}/3\mathbb{Z}(-4/3) + \mathbb{Z}/5\mathbb{Z}(-2/5)$  which agrees with  $q_{T_X}$  up to sign.

**5.2. Quotient family.** Denote the quotient elliptic surfaces with the 2-torsion section by  $Y$ . Then generically  $Y$  has singular fibres  $I_5, I_3, 8 \times I_2$  and  $\text{MW} \cong \mathbb{Z}/2\mathbb{Z}$ .

**Lemma 12.** *Generically  $Y$  has transcendental lattice  $T_Y \cong T_X(2)$ .*

*Proof.* By inspection of the singular fibres and torsion section  $\text{NS}(Y)$  has discriminant  $2^6 15$ . Then, as before, the push-forward embedding  $T_X(2) \hookrightarrow T_Y$  is an isometry.  $\square$

**5.3. Lattice enhancement.** We continue by enhancing  $\text{NS}(X)$  by a vector from the last summand of  $T_X$ . The analysis of the possible numbers which are represented primitively is a little more delicate. One can already observe this from the fact that each 2, 3 and 5 is represented, but neither 6, 10 or 15. This is related to the fact that the corresponding quadratic form gives the 2-torsion class in the class group  $Cl(-15)$ . This explains why a parity condition enters for the representations:

**Lemma 13.** *An integer  $-2N$  is represented primitively by  $\begin{pmatrix} -4 & -1 \\ -1 & -4 \end{pmatrix}$  if and only if*

- *$N$  is a product of an odd number of primes  $\equiv 2, 8 \pmod{15}$  or 15 times such a product;*
- *$N$  is 3 or 5 times a product of an even number of primes  $\equiv 2, 8 \pmod{15}$ .*

Enhancing  $\text{NS}$  by a primitive vector perpendicular to the vector from the lemma gives the transcendental lattice

$$T' = U^2 + \langle -2N \rangle.$$

The elliptic fibration is thus endowed with a section  $P$  of height  $2N/15$ .

**5.4. Proof of Theorem 1 (3).** Let  $N$  be as in Theorem 1 (3) (or equivalently Lemma 13). Let  $X'$  be a member of the subfamily of K3 surfaces given by the lattice enhancement of  $X$  corresponding to  $N$  as above. Then by assumption  $T_{X'}$  embeds primitively into  $U^2 + \langle -2N \rangle$ . Moreover the quotient surface has transcendental lattice  $T_{X'}(2)$  by Lemma 8 in conjunction with Lemma 13. Thus these surface realise a sandwiched Shioda–Inose structure.  $\square$

**5.5. Example:**  $N = 8$ . The elliptic fibration (10) ought to be equipped with a section  $P$  of height  $16/15$ . By the height pairing it suffices that  $P$  meets both  $I_{10}$  and  $I_6$  at  $\Theta_2$  (the second component) while not intersecting  $O$  as  $h(P) = 4 - \frac{2 \cdot 8}{10} - \frac{2 \cdot 4}{6} = \frac{16}{15}$ .

The present Weierstrass form (10) implies the shape of  $P$  to be  $(\alpha t^2, wt^2)$  with  $\alpha \in k^*$  and  $w \in k[t]$  of degree 2. But then the pair  $(\alpha, w)$  uniquely determines the polynomial  $a(t)$  in (10) by

$$w^2 = \alpha^3 t^2 + \alpha^2 a(t) + \alpha t.$$

Hence we obtain the 3-dimensional family of K3 surfaces with generically  $T = U^2 + \langle -16 \rangle$  (and the unirationality of the corresponding moduli space).

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